Proof of controllability of linear models

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Abstract

This brief note provides a proof of the controllability criterion presented and used in (Gu et al., 2015). The average energy to reach the states in the state space is also derived.

In Gu et al. (2015) the following linear model is used to simulate cerebral activity:

$$x(t) = A\ x(t-1) + B\ u(t), \tag{1}$$

where

- $x(t) \in \mathbb{R}^{N,1}$ is the state of the system at time t and N is the number of ROIs,
- $A \in \mathbb{R}^{N,N}$ is the adjacency matrix derived from DSI data,
- $B \in \mathbb{R}^{N,1}$ is the matrix encoding the control node that is used (i.e. the node used to control the system),
- $u(t) \in \mathbb{R}$ is the control input at time t.

Considering the following Gramian matrix

$$W = \sum_{k=0}^{\infty} A^k B B^T A^k, \tag{2}$$

the controllability criterion mentioned in the paper is that W must be invertible. Since W is positive semi definite, this criterion amounts to verify that the smallest eigenvalue of W is strictly greater than zero.

Controllability

The proof is as follows and assumes an initial condition $x(0) = 0$. Then the question of controllability amounts to verify whether there is a control strategy $\{u(0), u(1), \dots, u(\tau-1)\}$ to reach any desired target state $x(\tau)$. Mathematically, this target state is expressed as follows:

$$x(\tau) = \begin{bmatrix} B & AB & \dots & A^{\tau-1}B \end{bmatrix} \begin{bmatrix} u(\tau-1) \\ u(\tau-2) \\ \vdots \\ u(0) \end{bmatrix}. \tag{3}$$

Denoting by $C_\tau = \begin{bmatrix} B & AB & \dots & A^{\tau-1}B \end{bmatrix}$ ($C_\tau \in \mathbb{R}^{N,M}$), controllability as defined above amounts to ensure that C_τ is of full row rank since $x(\tau) \in \mathbb{R}^{N,1}$.

Since it can be shown that $rank(C_\tau C_\tau^T) = rank(C_\tau)$, imposing a full row rank to C_τ amounts to imposing full rank of $C_\tau C_\tau^T$. Hence we want the following matrix:

$$W_\tau = C_\tau C_\tau^T = \begin{bmatrix} B & AB & \dots & A^{\tau-1}B \end{bmatrix} \begin{bmatrix} B \\ AB \\ \vdots \\ A^{\tau-1}B \end{bmatrix} = \sum_{k=0}^{\tau-1} A^k B B^T A^k, \tag{4}$$

to be invertible. By taking the limit in $\tau \to \infty$ the condition (9) naturally comes, strict positivity of the eigenvalues of W coming from the positive semidefiniteness of W (i.e. we know all the eigenvalues are ≥ 0).

Average minimal energy

Trajectory of minimum energy (based on Li (2012))

Let us first derive the control strategy for reaching a specific destination x_{des} that corresponds to the minimal energy cost, where energy is defined as:

$$J(\mathbf{u}) = \frac{1}{2} \sum_{t=0}^{t=\tau-1} u(t)^T \cdot u(t) = \frac{1}{2} |\mathbf{u}|^2. \tag{5}$$

This is a constrained problem where $\mathbf{u}$ is the variable constrained by $x_{des} = C_\tau \cdot \mathbf{u}$. To solve this constrained optimization problem, the cost function is augmented:

$$J'(\mathbf{u}, \rho) = J(\mathbf{u}) + \rho^T (C_\tau \cdot \mathbf{u} - x_{des}) \tag{6}$$

Minimizing w.r.t. u and ρ leads to $x_{des} = C_\tau \cdot \mathbf{u}^*$ and $C_\tau^T \rho^* + \mathbf{u}^* = 0$ which has the following solution:

$$\rho^* = -(C_\tau C_\tau^T)^{-1} x_{des} = -W_\tau^{-1} x_{des} \tag{7}$$

$$\mathbf{u}^* = C_\tau^T W_\tau^{-1} x_{des}, \tag{8}$$

and the optimal cost of control is:

$$J(\mathbf{u}^*) = x_{des}^T W_\tau^{-1} C_\tau C_\tau^T W_\tau^{-1} x_{des} = x_{des}^T W_\tau^{-1} x_{des}. \tag{9}$$

It is interesting to note that the destinations corresponding to the eigenvectors associated with the lowest eigenvalues require the most energy to be reached. Indeed, let x_{des} be v_i, the i^{th} eigenvector of W (and hence of W^{-1}). Then the minimal energy to reach this destination is:

$$E(v_i) = v_i^T W^{-1} v_i = v_i^T \frac{1}{\lambda_i} v_i = \frac{1}{\lambda_i} \tag{10}$$

where λ_i is the i^{th} eigenvalue of W (hence $\frac{1}{\lambda_i}$ is the i^{th} eigenvalue of W^{-1}).

Measure of average controllability

It is claimed in Gu et al. (2015) that the average (minimal) energy required to reach all the states is proportional to the trace of W^{-1}. I didn't find a rigorous proof of this but intuition can be obtained for this from equations (9) and (13). Indeed, by considering the spectral decomposition of the destination $x_{des} = \sum_{i=1}^{N} \alpha_i v_i$ where, as here above, v_i is the i^{th} eigenvector of W. The energy to reach x_{des} can be formulated as:

$$E(x_{des}) = \sum_{i=1}^{N} \alpha_i v_i^T \cdot W^{-1} \cdot \sum_{i=1}^{N} \alpha_i v_i \quad (11)$$

$$= \sum_{i=1}^{N} \alpha_i v_i^T \sum_{i=1}^{N} \alpha_i \frac{1}{\lambda_i} v_i \quad (12)$$

$$= \sum_{i=1}^{N} \alpha_i^2 \frac{1}{\lambda_i} \quad (13)$$

To derive the (minimal) energy to reach x_{des}, in average over all possibilities of x_{des}, the coefficients α_i are taken to vary uniformely and independently in a certain set, leading to $\mathbb{E}_{x_{des}}[E(x_{des})] \propto \sum_{i=1}^{N} \frac{1}{\lambda_i} = \mathrm{Tr}(W^{-1})$.

In Gu et al. (2015) they chose to use $\mathrm{Tr}(W)$ (instead of $\mathrm{Tr}(W^{-1})$) to characterize average controllability because due to the (very) small eigenvalues of W its inversion is not reliable. There is obviously a link between $\mathrm{Tr}(W)$ and $\mathrm{Tr}(W^{-1})$ but this last part is not clear to me, and not well motivated in their supplementary material. One thing is that they associate $\mathrm{Tr}(W)$ to *controllability* and $\mathrm{Tr}(W^{-1})$ to *energy*, which seems to make sense. However, if W has both very large and very small eigenvalues, the link between the two traces might not be straightforward. To be studied in more details.

References

Gu, S., F. Pasqualetti, M. Cieslak, Q. K. Telesford, B. Y. Alfred, A. E. Kahn, J. D. Medaglia, J. M. Vettel, M. B. Miller, S. T. Grafton, et al.
2015. Controllability of structural brain networks. *Nature communications*, 6.

Li, P. Y.
2012. *Controllability and Observability*.